Maximization of capacity and l_p norms for some product channels.

Christopher King
Department of Mathematics
Northeastern University
Boston, MA 02115
king@neu.edu

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Abstract

It is conjectured that the Holevo capacity of a product channel $\Omega \otimes \Phi$ is achieved when product states are used as input. Amosov, Holevo and Werner have also conjectured that the maximal l_p norm of a product channel is achieved with product input states. In this paper we establish both of these conjectures in the case that Ω is arbitrary and Φ is a CQ or QC channel (as defined by Holevo). We also establish the Amosov, Holevo and Werner conjecture when Ω is arbitrary and either Φ is a qubit channel and $p = 2$, or Φ is a unital qubit channel and p is integer. Our proofs involve a new conjecture for the norm of an output state of the half-noisy channel $I \otimes \Phi$, when Φ is a qubit channel. We show that this conjecture in some cases also implies additivity of the Holevo capacity.

1 Introduction

A quantum channel is the mathematical description of a device which stores and transmits quantum states. Much work has been devoted to the study of particular quantum channels with highly non-classical properties, and also to general questions such as the information capacity of classes of channels. In this paper we will consider some problems of the second type, concerning additivity and multiplicativity properties that are believed to hold for all product channels.

The basic components of a quantum channel are a Hilbert space $\mathcal{H}$ and a noise operator Φ. The quantum states are positive operators on $\mathcal{H}$, with trace equal to one. The noise operator Φ is a completely positive, trace-preserving map which acts on the set of states. Positivity means that Φ is a positive operator on $B(\mathcal{H})$ (the algebra of bounded operators on $\mathcal{H}$). Complete positivity means that the map $I \otimes \Phi$ is also a positive operator on $B(\mathbf{C}^K \otimes \mathcal{H})$ for every K.

When the channel $(\mathcal{H}, \Phi)$ is used to store or transmit information, it is assumed that the information is encoded as a state on the product space $\mathcal{H}^{\otimes n}$ for some n, and that the noise acts on this state through the product operator $\Phi^{\otimes n}$, thereby mimicking the action of a memoryless channel in classical information theory. The basic properties of such quantum memoryless channels have been studied by many authors [3], [5], [8], [9], [16]. One outstanding problem is to determine the ultimate rate at which classical information can be transmitted through this channel, when no prior entanglement is available between sender and receiver. The protocol that achieves this capacity may require messages to be encoded using entangled states and/or decoded using collective measurements. It is conjectured that this ultimate capacity is given by the well-known Holevo bound [8]

$$C_{\mathrm{Holv}}(\Phi) = \sup_{\pi,\,\rho} \Big[S\big(\sum \pi_i \Phi(\rho_i)\big) - \sum \pi_i S(\Phi(\rho_i)) \Big], \qquad (1)$$

where $S(\rho) = -\mathrm{Tr}\rho \log \rho$ is the von Neumann entropy, and the sup runs over all probability distributions $\{\pi_i\}$ and collections of states $\{\rho_i\}$ on $\mathcal{H}$. This capacity conjecture is equivalent to the statement that there is no benefit gained when entangled states are used to encode messages for transmission through a quantum channel. As shown by Holevo [8] and Schumacher-Westmoreland [16], the ultimate rate for information transmission using non-entangled coding states is exactly C_{Holv}. Thus the capacity conjecture is implied by the *additivity*

conjecture for C_{Holv}, which states that for any channels Ω and Φ

$$C_{\text{Holv}}(\Omega \otimes \Phi) = C_{\text{Holv}}(\Omega) + C_{\text{Holv}}(\Phi) \tag{2}$$

Although the equality (2) has been shown in some special cases [1], [4], [9], [11], [16], it remains a challenging problem to prove this result for a general pair of channels (Ω, Φ). Amosov, Holevo and Werner introduced a related conjecture, concerning the noncommutative l_p norm of output states from a product channel [1] (this norm is defined below). In this paper we report progress toward establishing these conjectures for some special product channels, namely the cases when Ω is arbitrary and either (i) Φ is a CQ or QC channel (these are defined below), or (ii) Φ is a qubit channel. In the first case we establish both conjectures. In the second case we establish the Amosov, Holevo and Werner conjecture for integer values of p. A principal ingredient in our proof in the second case is a new bound concerning the l_p norm of the output from a "half-noisy" channel $I \otimes \Phi$, for integer values of p. We conjecture that this bound holds for all $p \geq 1$, and we show that in some cases this conjecture implies additivity of the Holevo bound (2).

The paper is organised as follows. Section 2 contains a precise statement of the results, and the conjectured bound for half-noisy channels. In section 3 we review the relation of relative entropy and the Holevo bound. In sections 4 and 5 we prove the results for CQ and QC channels. Then in section 6 we prove the results for qubit channels, and in section 7 we prove the Corollaries of our new conjecture. In section 8 we give a summary and overview of the results in the paper. Finally the Appendix contains a proof by Lieb and Ruskai of a special case of the conjecture.

2 Statement of results

The noncommutative l_p norm of a matrix A is defined by

$$||A||_p = (\text{Tr}|A|^p)^{\frac{1}{p}} = \left[\text{Tr}(A^*A)^{\frac{p}{2}}\right]^{\frac{1}{p}} \tag{3}$$

The corresponding maximal l_p norm for a positive map Φ on $B(\mathcal{H})$ is

$$\nu_p(\Phi) = \sup_{\rho} ||\Phi(\rho)||_p \tag{4}$$

where the sup runs over states in $\mathcal{H}$ (this quantity was introduced in [1], where it was called the 'maximal output purity' of the channel). It is always true that for any maps Ω and Φ, and any $p \geq 1$

$$\nu_p(\Omega \otimes \Phi) \geq \nu_p(\Omega)\,\nu_p(\Phi) \tag{5}$$

The multiplicativity conjecture of [1] states that for any completely positive trace-preserving maps Ω and Φ, and for all $p \geq 1$,

$$\nu_p(\Omega \otimes \Phi) = \nu_p(\Omega)\,\nu_p(\Phi) \tag{6}$$

Equality always holds in (6) for $p = 1$. It has been shown in several different ways that (6) holds for all $p \geq 1$ and all Ω when $\Phi = I$ [1], [6], [17]. Recently, it has been shown that (6) holds when both Ω and Φ are depolarizing channels, and p is integer [2]. In this paper we provide some further examples where it holds.

The first case we consider involves the CQ and QC channels introduced by Holevo [9], so we recall their definitions now. Let $\{X_b\}$ be a POVM on $\mathcal{H}$ (so $X_b \geq 0$ and $\sum X_b = I$) and let $\{Q_b\}$ be any collection of states. Then we can define a channel Φ by the formula

$$\Phi(\rho) = \sum \mathrm{Tr}(\rho\, X_b)\, Q_b \tag{7}$$

Holevo considered two special cases of (7). First, if $\{X_b = |e_b\rangle\langle e_b|\}$ are projections onto an orthonormal basis $\{|e_b\rangle\}$ in $\mathcal{H}$, then (7) is called a CQ channel. Second, if $\{Q_b = |e_b\rangle\langle e_b|\}$, then (7) is called a QC channel. Holevo proved the additivity result (2) when $\Omega = \Phi$ is either a CQ or QC channel. Our first result generalises this by allowing an arbitrary channel Ω.

Theorem 1 *Let Φ be a* CQ *or* QC *channel. Then for any completely positive trace-preserving map Ω, l_p-multiplicativity (6) holds for all $p \geq 1$, and Holevo additivity (2) holds.*

For our second set of results we restrict to channels on a two-dimensional Hilbert space. For brevity of notation we will say that Φ is a *qubit map* if it is a completely positive trace-preserving map on $\mathcal{B}(\mathbf{C}^2)$.

Theorem 2 *Let Φ be a qubit channel. Then the equality (6) holds for $p = 2$, that is $\nu_2(\Omega \otimes \Phi) = \nu_2(\Omega)\ \nu_2(\Phi)$ for all channels Ω.*

In order to state the next result we need to recall the classification of qubit maps. Any qubit map Φ can be represented by a real 4×4 matrix with respect to the basis $I, \sigma_1, \sigma_2, \sigma_3$, where σ_i are the Pauli matrices. In [11] it was explained that by using independent unitary transformations in its domain and range, this matrix can be put into the following form:

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{pmatrix} \tag{8}$$

This form makes it easy to see how Φ acts on the Bloch sphere. The sphere is first compressed to an ellipsoid with semi-major axes $|\lambda_1|, |\lambda_2|, |\lambda_3|$, and is then translated by the vector $\mathbf{t} = (t_1, t_2, t_3)$. There are constraints on the allowed values of these six parameters (coming from the requirements that Φ be completely positive and trace-preserving), and these constraints have been fully worked out in [15]. If $t_i = 0$ for $i = 1, 2, 3$ then $\Phi(I) = I$, in which case Φ is a *unital* qubit map.

Our next result requires a slightly stronger condition on the map Φ, which we now state in terms of these parameters:

$$\text{if} \quad |\lambda_i| < |\lambda_j| < |\lambda_k| \quad \text{then} \quad t_i t_j = 0 \tag{9}$$

This condition can be stated in words as follows: the ellipsoid may be translated only in directions lying in the two planes that are perpendicular to its two smaller axes (if any two axes have equal length, there is no restriction).

Theorem 3 *Let Φ be a qubit channel satisfying the condition (9). Then l_p-multiplicativity (6) holds for all integer p, that is $\nu_p(\Omega \otimes \Phi) = \nu_p(\Omega)\ \nu_p(\Phi)$ for all channels Ω and all integers p.*

The proofs of Theorem 2 and Theorem 3 make use of a bound for the l_p norm of the output state from the half-noisy channel $I \otimes \Phi$. We believe that this bound holds for all $p \geq 1$, however we can prove it only for the cases listed in the Theorems. So we state the general bound as a conjecture.

Conjecture 4 *Let Φ be a qubit channel, and let $M \geq 0$ be a $2K \times 2K$ matrix. Write M in the form*

$$M = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}, \tag{10}$$

where X, Y and Z are $K \times K$ matrices. Then for all $p \geq 1$

$$||(I \otimes \Phi)(M)||_p \leq \nu_p(\Phi) \, (||X||_p + ||Z||_p) \tag{11}$$

This conjecture has several important consequences, which we list in the next three Corollaries. In particular, the first Corollary shows that Conjecture 4 implies Theorems 2 and 3.

Corollary 5 *Let Φ be a qubit channel, and suppose that (11) holds for all positive $2K \times 2K$ matrices M, for some $p \geq 1$. Then for any completely positive map Ω on $B(\mathbf{C}^K)$, l_p-multiplicativity (6) holds for the same value of p.*

In Section 5 we will prove that (11) holds for all qubit maps Φ when $p = 2$, and also for the cases listed in Theorem 3. Combining this with Corollary 5 will prove Theorems 2 and 3.

Our next result concerns the additivity of minimal entropy. The minimal entropy of a completely positive trace-preserving map Φ is defined by

$$S_{\min}(\Phi) = \inf_{\rho} S(\Phi(\rho)) \tag{12}$$

The additivity of minimal entropy is the statement that

$$S_{\min}(\Omega \otimes \Phi) = S_{\min}(\Omega) + S_{\min}(\Phi) \tag{13}$$

Corollary 6 *Let Φ be a qubit channel, and suppose that (11) holds for all positive $2K \times 2K$ matrices M, and for all $p \in [1, s)$ for some $s > 1$. Then for any completely positive map Ω on $B(\mathbf{C}^K)$, additivity of minimal entropy (13) holds.*

For our last corollary, recall that a map Φ is unital if $\Phi(I) = I$, which means roughly that Φ leaves unchanged the "noisiest" state through the channel.

Corollary 7 *Let Φ be a unital qubit channel, and suppose that (11) holds for all positive $2K \times 2K$ matrices M, and for all $p \in [1, s)$ for some $s > 1$. Then for any completely positive trace-preserving map Ω on $B(\mathbf{C}^K)$, Holevo additivity (2) holds.*

Remarks.

1) There are two special cases where it is easy to verify Conjecture 4. First, if M is a one-dimensional projection then the right side of (11) becomes $\nu_p(\Phi)\ \mathrm{Tr}(M)$, and then the result follows immediately from the definition (4). Second, suppose that Φ is the identity map, so $\nu_p(\Phi) = 1$. Define the projections

$$P_0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \tag{14}$$

Then convexity of the l_p norm for $p \geq 1$ implies that

$$||M||_p = ||M^{1/2}(P_0 + P_1)M^{1/2}||_p \leq ||M^{1/2} P_0 M^{1/2}||_p + ||M^{1/2} P_1 M^{1/2}||_p \tag{15}$$

Furthermore for any matrix A, the matrices AA^* and A^*A have the same spectrum, so we deduce that

$$||M||_p \leq ||P_0 M P_0||_p + ||P_1 M P_1||_p = ||X||_p + ||Z||_p \tag{16}$$

(this derivation is a special case of a more general result for POVM's which is described in [7]).

2) Lieb and Ruskai have recently established Conjecture 4, eq. (11) for a depolarizing channel in the special case $X = Z$, for all $p \geq 1$. Recall that the depolarizing channel is described by the parameter values $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, and $t_1 = t_2 = t_3 = 0$, so that in this case the bound (11) becomes

$$\Big|\Big| \begin{pmatrix} X & \lambda Y \\ \lambda Y^* & X \end{pmatrix} \Big|\Big|_p \leq \nu_p(\Phi)\ (2\,||X||_p) \tag{17}$$

where

$$\nu_p(\Phi) = \Big[\Big(\frac{1+\lambda}{2}\Big)^p + \Big(\frac{1-\lambda}{2}\Big)^p \Big]^{1/p} \tag{18}$$

Their proof appears as an Appendix to this paper.

3) Theorem 3 was proved in [11] for unital maps in the case $p = \infty$, and our proof here extends this result to all integer values of p (and to a larger class of maps). The class of qubit maps which satisfy (9) includes all unital qubit maps and many non-unital maps. In particular, our proof applies to any extreme point in the set of qubit maps (this refers to recent work in [15], and we discuss it more fully in section 3).

4) To prove Corollaries 6 and 7 we need only the derivative of (11) at $p = 1$, which we now state as a separate bound. Assume that M has the form (10) with $\mathrm{Tr}(M) = 1$, and define the states

$$\xi = \frac{1}{\mathrm{Tr}X}\, X, \qquad \zeta = \frac{1}{\mathrm{Tr}Z}\, Z \tag{19}$$

Then taking the derivative of (11) at $p = 1$ gives

$$S((I \otimes \Phi)(M)) \geq S_{\min}(\Phi) + \mathrm{Tr}(X)S(\xi) + \mathrm{Tr}(Z)S(\zeta) \tag{20}$$

5) When Ω and Φ are both unital qubit maps, the additivity result (2) follows immediately from the additivity of minimal entropy (13), as was discussed in [11]. This is also true if $\Omega = \Phi_1 \otimes \cdots \otimes \Phi_n$ is a product of unital qubit maps. Additivity of Holevo capacity (2) for the 'half-noisy' case $\Phi = I$ was proved by Schumacher and Westmoreland [17], and their analysis underlies our proof of Corollary 7.

3 Relative entropy and the Holevo bound

The Holevo bound (1) can be re-expressed in terms of relative entropy in several ways (see for example the discussion in [12]). Here we will follow the approach of Ohya, Petz and Watanabe [14] and Schumacher and Westmoreland [17], who express (1) as an optimization of relative entropy.

Let Φ be a channel, and let $\mathcal{E} = \{\pi_i, \rho_i\}$ be an ensemble of input states for the channel. Define

$$\chi(\Phi; \mathcal{E}) = S\Big(\sum \pi_i \Phi(\rho_i)\Big) - \sum \pi_i S(\Phi(\rho_i)) \tag{21}$$

Following the notation of [17], the Holevo capacity of the channel is denoted

$$\chi^*(\Phi) = C_{\mathrm{Holv}}(\Phi) = \sup_{\mathcal{E}} \chi(\Phi; \mathcal{E}) \tag{22}$$

As shown in [17] there is an ensemble which achieves this supremum. The ensemble may not be unique, however its average input state is unique. We let $\rho^* = \sum \pi_i \rho_i$ denote this optimal average input state.

The relative entropy of a state ω with respect to a state ρ is defined by

$$S(\omega \,|\, \rho) = \mathrm{Tr}\, \omega \,(\log \omega - \log \rho) \tag{23}$$

Relative entropy is non-negative: $S(\omega \,|\, \rho) \geq 0$, with equality if and only if $\omega = \rho$. There is a useful characterization of the capacity $\chi^*(\Phi)$ in terms of relative entropy, namely

$$\chi^*(\Phi) = \inf_{\rho} \sup_{\omega} S(\Phi(\omega) \,|\, \Phi(\rho)) \tag{24}$$

This result was derived in [14] and also in [17]. For our purposes it is convenient to restate it as follows: for any state ρ,

$$\chi^*(\Phi) \leq \sup_{\omega} S(\Phi(\omega) \,|\, \Phi(\rho)) \tag{25}$$

and equality holds in (25) if and only if $\rho = \rho^*$.

Our goal is the additivity result (2). By restricting to product states it is clear that

$$\chi^*(\Omega) + \chi^*(\Phi) \leq \chi^*(\Omega \otimes \Phi) \tag{26}$$

So to establish (2) it is sufficient to prove the bound

$$\chi^*(\Omega \otimes \Phi) \leq \chi^*(\Omega) + \chi^*(\Phi) \tag{27}$$

For a channel Φ, denote the optimal average *output* state by

$$\rho_\Phi := \Phi(\rho^*) \tag{28}$$

Then (25) implies that

$$\chi^*(\Omega \otimes \Phi) \leq \sup_{\tau} S\Big((\Omega \otimes \Phi)(\tau) \,|\, \rho_\Omega \otimes \rho_\Phi\Big) \tag{29}$$

Therefore in order to prove (27), and hence (2), it is sufficient to show that for any state τ,

$$S\Big((\Omega \otimes \Phi)(\tau) \,|\, \rho_\Omega \otimes \rho_\Phi\Big) \leq \chi^*(\Omega) + \chi^*(\Phi) \tag{30}$$

4 Proof for CQ channel

Let Φ be a CQ channel on $B(\mathbf{C}^N)$, so that

$$\Phi(\rho) = \sum \mathrm{Tr}(\rho X_b) Q_b, \tag{31}$$

where $\{X_b\}$ are one-dimensional orthogonal projections. It follows that for all $b = 1, \dots, N$,

$$Q_b = \Phi(X_b) \tag{32}$$

Let Ω be a completely positive map on $B(\mathbf{C}^K)$. Then for any state τ in $B(\mathbf{C}^K \otimes \mathbf{C}^N)$,

$$(\Omega \otimes \Phi)(\tau) = \sum \Omega\Big(\mathrm{Tr}_2((I \otimes X_b)\, \tau)\Big) \otimes Q_b \tag{33}$$

where Tr_2 is the trace over the second factor. For each $b = 1, \dots, N$ let

$$n_b = \mathrm{Tr}((I \otimes X_b)\, \tau), \tag{34}$$

and define the state

$$\tau_b = \frac{1}{n_b}\ \mathrm{Tr}_2((I \otimes X_b)\, \tau) \tag{35}$$

Then (33) can be written

$$(\Omega \otimes \Phi)(\tau) = \sum n_b\, \Omega(\tau_b) \otimes Q_b = \sum n_b\, \Omega(\tau_b) \otimes \Phi(X_b) \tag{36}$$

where in the second equality we used (32).

Turning first to the l_p norm result, it follows from (36) and the definition (4) that

$$||(\Omega \otimes \Phi)(\tau)||_p \leq \sum n_b\, \nu_p(\Omega)\, \nu_p(\Phi) = \nu_p(\Omega)\, \nu_p(\Phi) \tag{37}$$

and this proves (6).

Turning next to the channel capacity result, we will prove that (30) holds. Indeed (36) implies that

$$S\Big((\Omega \otimes \Phi)(\tau) \,|\, \rho_\Omega \otimes \rho_\Phi\Big) \leq \sum n_b \left[S\Big(\Omega(\tau_b) \,|\, \rho_\Omega\Big) + S\Big(\Phi(X_b) \,|\, \rho_\Phi\Big)\right] \tag{38}$$

where we used the additivity of relative entropy for product states. Now (24) implies

$$S\Big(\Omega(\tau_b) \,|\, \rho_\Omega\Big) \leq \chi^*(\Omega), \qquad S\Big(\Phi(X_b) \,|\, \rho_\Phi\Big) \leq \chi^*(\Phi) \tag{39}$$

which proves the result.

5 Proof for QC channel

Let Φ be a QC channel, so that

$$\Phi(\rho) = \sum \text{Tr}(\rho X_b) Q_b, \tag{40}$$

where $\{Q_b\}$ are one-dimensional orthogonal projections. For any state τ,

$$\begin{aligned}(\Omega \otimes \Phi)(\tau) &= \sum \Omega(\text{Tr}_2(I \otimes X_b)\tau) \otimes Q_b \\ &= \sum n_b \, \Omega(\tau_b) \otimes Q_b\end{aligned} \tag{41}$$

where we use the definitions (34) and (35). Now define

$$\theta = \text{Tr}_1(\tau), \tag{42}$$

then it follows that

$$n_b = \text{Tr}(\theta \, X_b) \tag{43}$$

and (41) can be written as

$$(\Omega \otimes \Phi)(\tau) = \sum \Omega(\tau_b) \otimes (\text{Tr}(\theta \, X_b) \, Q_b) \tag{44}$$

First we prove the bound for the l_p norm. Using the fact that $\{Q_b\}$ are orthogonal projections, we get

$$\text{Tr}|(\Omega \otimes \Phi)(\tau)|^p = \sum \text{Tr}|\Omega(\tau_b)|^p \; (\text{Tr}(\theta \, X_b))^p \tag{45}$$

The definition of the l_p norm implies that for any positive matrix A,

$$||\Omega(A)||_p \leq \nu_p(\Omega) \; \text{Tr}(A) \tag{46}$$

and hence (45) implies that

$$\text{Tr}|(\Omega \otimes \Phi)(\tau)|^p \leq (\nu_p(\Omega))^p \; \sum (\text{Tr}(\theta \, X_b))^p \tag{47}$$

Furthermore, from (40) it follows that

$$\text{Tr}|\Phi(\theta)|^p = \sum [\text{Tr}(\theta X_b)]^p \tag{48}$$

Combining (47) and (48) and taking the p^{th} root gives

$$||\Omega \otimes \Phi(\tau)||_p \leq \nu_p(\Omega)\ ||\Phi(\theta)||_p \leq \nu_p(\Omega)\ \nu_p(\Phi) \tag{49}$$

which then proves the result.

Turning now to the additivity of the channel capacity, we will again establish the bound (30). We claim that the following identity holds:

$$S\Big((\Omega \otimes \Phi)(\tau) \,|\, \rho_\Omega \otimes \rho_\Phi\Big) = \sum \mathrm{Tr}(\theta\, X_b) S\Big(\Omega(\tau_b) \,|\, \rho_\Omega\Big) + S\Big(\Phi(\theta) \,|\, \rho_\Phi\Big) \tag{50}$$

From the result (25) it follows that

$$S\Big(\Phi(\theta) \,|\, \rho_\Phi\Big) \leq \chi^*(\Phi), \qquad S\Big(\Omega(\tau_b) \,|\, \rho_\Omega\Big) \leq \chi^*(\Omega) \tag{51}$$

Therefore (50) implies

$$S\Big((\Omega \otimes \Phi)(\tau) \,|\, \rho_\Omega \otimes \rho_\Phi\Big) \leq \sum \mathrm{Tr}(\theta\, X_b)\chi^*(\Omega) + \chi^*(\Phi) = \chi^*(\Omega) + \chi^*(\Phi) \tag{52}$$

and this proves the result.

So it remains to verify the identity (50). This follows easily from the definition of relative entropy, and the fact that $\{Q_b\}$ are orthogonal projections.

6 Proofs for qubit channels

In this section we prove Theorems 2 and 3. We do this by establishing the bound (11), and then using Corollary 5, which will be proved in the next section.

Let Φ be a qubit map, and assume that bases have been chosen in its domain and range so that it has the form (8). Clearly, the maximal l_p norm of Φ is invariant under permutations of the three coordinates. It is also invariant under the following symmetry operations.

Lemma 8 *For every p, $\nu_p(\Phi)$ is invariant if the signs of any two of $(\lambda_1, \lambda_2, \lambda_3)$ are reversed, or if the signs of any two of (t_1, t_2, t_3) are reversed.*

The proof is easy: first notice that conjugation by σ_1 in the domain of Φ switches the signs of λ_2, λ_3 without any other changes, and similarly for conjugation by σ_2 and σ_3. Then notice that simultaneous conjugation by σ_1 in

both the domain and range of Φ switches the signs of t_2, t_3 without any other changes, and similarly for σ_2 and σ_3.

As a consequence, we will assume henceforth without loss of generality that

$$t_1 \geq 0, t_2 \geq 0, \quad \text{and} \quad \lambda_1 \geq \lambda_2 \geq 0 \tag{53}$$

Our first goal is to establish Conjecture 4 for $p = 2$, for any map Φ. We rewrite (10) more fully as

$$M = \begin{pmatrix} X & Y_1 - iY_2 \\ Y_1 + iY_2 & Z \end{pmatrix} \tag{54}$$

where $X > 0$, $Z > 0$ and Y_1, Y_2 are hermitian. Let $W = (X + Z)/2$. Then using the special form (8) we get

$$(I \otimes \Phi)(M) = \tag{55}$$
$$\begin{pmatrix} c_{++}X + c_{-+}Z & (t_1 W + \lambda_1 Y_1) - i(t_2 W + \lambda_2 Y_2) \\ (t_1 W + \lambda_1 Y_1) + i(t_2 W + \lambda_2 Y_2) & c_{--}X + c_{+-}Z \end{pmatrix}$$

where

$$\begin{aligned} c_{++} &= (1 + \lambda_3 + t_3)/2, \qquad & c_{-+} &= (1 - \lambda_3 + t_3)/2 \\ c_{+-} &= (1 + \lambda_3 - t_3)/2, \qquad & c_{--} &= (1 - \lambda_3 - t_3)/2 \end{aligned} \tag{56}$$

Note that since $M \geq 0$ and Φ is a qubit map, it follows that $(I \otimes \Phi)(M) \geq 0$ for all choices of X and Z. Hence the four coefficients in (56) are positive, for all allowed values of t_3 and λ_3.

We consider first the case that $p = 2$, and Φ is any qubit map. Taking the trace of the square of (55) gives

$$\begin{aligned} \text{Tr}|(I \otimes \Phi)(M)|^2 &= \text{Tr}(c_{++}X + c_{-+}Z)^2 + \text{Tr}(c_{+-}X + c_{--}Z)^2 \\ &+ 2\text{Tr}(t_1 W + \lambda_1 Y_1)^2 + 2\text{Tr}(t_2 W + \lambda_2 Y_2)^2 \end{aligned} \tag{57}$$

Define

$$x = ||X||_2, \quad z = ||Z||_2, \quad y_1 = ||Y_1||_2, \quad y_2 = ||Y_2||_2 \tag{58}$$

Then using the Cauchy-Schwarz inequality for the Hilbert-Schmidt norm, and our positivity condition (53) we get

$$\begin{aligned}\mathrm{Tr}|(I \otimes \Phi)(M)|^2 &\leq (c_{++}x + c_{-+}z)^2 + (c_{+-}x + c_{--}z)^2 \\ &+ 2\Big(t_1 \frac{(x+z)}{2} + \lambda_1 y_1\Big)^2 + 2\Big(t_2 \frac{(x+z)}{2} + \lambda_2 y_1\Big)^2\end{aligned} \tag{59}$$

Define the 2×2 matrix

$$m = \begin{pmatrix} x & y_1 - iy_2 \\ y_1 + iy_2 & z \end{pmatrix} \tag{60}$$

Then (59) can be re-written as

$$||I \otimes \Phi(M)||_2 \leq ||\Phi(m)||_2 \tag{61}$$

The positivity of M implies that

$$\mathrm{Tr}|Y_1 - iY_2|^2 = y_1^2 + y_2^2 \leq xz, \tag{62}$$

and hence that m is positive. Therefore

$$||(I \otimes \Phi)(M)||_2 \leq \nu_2(\Phi)\mathrm{Tr}(m) = \nu_2(\Phi)(x+z) = \nu_2(\Phi)(||X||_2 + ||Z||_2) \tag{63}$$

which establishes (11) for $p = 2$, and hence by Corollary 5 proves Theorem 2.

In order to prove Theorem 3 we will assume that the condition (9) is satisfied. Without loss of generality, this condition can be rewritten as follows:

$$t_1 \geq 0 \quad \text{and} \quad t_2 = 0 \quad \text{and} \quad \lambda_1 \geq \lambda_2 \geq 0. \tag{64}$$

To see this, suppose first that $|\lambda_i| \neq |\lambda_j|$ for any i, j. Then the condition (9) implies that at least one of the t_i is zero, and also that the corresponding $|\lambda_i|$ is not the largest. Hence by permuting coordinates we can arrange that $t_2 = 0$ and that $|\lambda_1| > |\lambda_2|$. By switching signs of pairs of parameters we can then re-state (9) as (64). Suppose now that $|\lambda_i| = |\lambda_j|$ for some i, j. By permuting coordinates we can assume that $|\lambda_1| = |\lambda_2|$, and by changing signs that $\lambda_1 = \lambda_2 \geq 0$. This allows a further symmetry transformation, namely we can conjugate by a unitary matrix $U = e^{i\theta\sigma_3}$ in the range of Φ without changing $\nu_p(\Phi)$. With such a conjugation we can set $t_2 = 0$, and then the condition (64) again holds.

The condition (64) is clearly satisfied for all unital maps, since in that case $t_i = 0$ for all i. It is also satisfied by all maps in the closure of the set of extreme points of the (convex) set of qubit maps. This fact follows from Theorem 4 in [15], where it was shown that all such maps have only one of the parameters t_1, t_2, t_3 being non-zero.

In order to prove (11), we re-write (55) as

$$\begin{aligned}(I \otimes \Phi)(M) &= \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \\ &= R_{11} \otimes E_{11} + R_{12} \otimes E_{12} + R_{21} \otimes E_{21} + R_{22} \otimes E_{22}\end{aligned} \tag{65}$$

where E_{ij} is the 2×2 matrix with 1 in position (i, j) and 0 elsewhere, and where

$$\begin{aligned} R_{11} &= c_{++}X + c_{-+}Z, \\ R_{12} &= (t_1 W + \lambda_1 Y_1) - i\lambda_2 Y_2, \\ R_{21} &= (t_1 W + \lambda_1 Y_1) + i\lambda_2 Y_2, \\ R_{22} &= c_{--}X + c_{+-}Z \end{aligned} \tag{66}$$

(we have used the condition (64) to set $t_2 = 0$).

For integer p we can evaluate $\mathrm{Tr}|(I \otimes \Phi)(M)|^p$ by multiplying the right side of (65) with itself p times, and taking the trace with respect to a product basis $e_i \otimes f_j$ where $\{e_i\}$ span $\mathbf{C}^K$ and f_1, f_2 span $\mathbf{C}^2$. The result is

$$\mathrm{Tr}|(I \otimes \Phi)(M)|^p = \sum \mathrm{Tr}[E_{i_1 j_1} E_{i_2 j_2} \ldots E_{i_p j_p}]\ \mathrm{Tr}[R_{i_1 j_1} R_{i_2 j_2} \ldots R_{i_p j_p}], \tag{67}$$

where the sum runs over all indices $i_1, j_1, \ldots, i_p, j_p = 1, 2$. The coefficient $\mathrm{Tr}[E_{i_1 j_1} E_{i_2 j_2} \ldots E_{i_p j_p}]$ in each of these terms is non-negative, since the matrices $\{E_{ij}\}$ are all non-negative. Furthermore, repeated application of Hölder's inequality shows that

$$|\mathrm{Tr} A_1 A_2 \ldots A_p| \leq ||A_1||_p\ ||A_2||_p\ \ldots\ ||A_p||_p \tag{68}$$

for any product of p matrices. Hence the sum in (67) is bounded above by

$$\mathrm{Tr}|(I \otimes \Phi)(M)|^p \leq \sum \mathrm{Tr}[E_{i_1 j_1} E_{i_2 j_2} \ldots E_{i_p j_p}]\ ||R_{i_1 j_1}||_p ||R_{i_2 j_2}||_p \ldots ||R_{i_p j_p}||_p \tag{69}$$

We define the 2×2 matrix

$$m' = \begin{pmatrix} x' & y' \\ y' & z' \end{pmatrix} \tag{70}$$

where now

$$x' = ||X||_p, \quad z' = ||Z||_p, \quad y' = ||Y_1 - iY_2||_p \tag{71}$$

The matrix m' is positive. This can be seen most easily by noting that the positivity of M implies that $Y_1 - iY_2 = \sqrt{X}\, T\, \sqrt{Z}$ where T is a contraction [15], and hence by Hölder's inequality $y' \leq \sqrt{x'\, z'}$. Applying the map Φ gives

$$\begin{aligned} \Phi(m') &= [c_{++}x' + c_{-+}z'] \otimes E_{11} + [t_1(x'+z')/2 + \lambda_1 y'] \otimes E_{12} \\ &\quad + [t_1(x'+z')/2 + \lambda_1 y'] \otimes E_{21} + [c_{--}x' + c_{+-}z'] \otimes E_{22} \end{aligned} \tag{72}$$

Applying the same method to evaluate $\mathrm{Tr}|\Phi(m')|^p$ gives

$$\mathrm{Tr}|\Phi(m')|^p = \sum \mathrm{Tr}[E_{i_1 j_1} E_{i_2 j_2} \ldots E_{i_p j_p}]\ r_{i_1 j_1} r_{i_2 j_2} \ldots r_{i_p j_p} \tag{73}$$

where

$$\begin{aligned} r_{11} &= c_{++}\, x' + c_{-+}\, z', \\ r_{12} &= r_{21} = t_1(x'+z')/2 + \lambda_1 y', \\ r_{22} &= c_{--}\, x' + c_{+-}\, z' \end{aligned} \tag{74}$$

We now claim that

$$\mathrm{Tr}|(I \otimes \Phi)(M)|^p \leq \mathrm{Tr}|\Phi(m')|^p \tag{75}$$

If we assume for the moment that (75) is valid, then it implies

$$||(I \otimes \Phi)(M)||_p \leq ||\Phi(m')||_p \leq \nu_p(\Phi)\ \mathrm{Tr}(m') \leq \nu_p(\Phi)\ (x' + z') \tag{76}$$

This proves (11), which by Corollary 5 implies Theorem 3.

So it sufficient to demonstrate (75). From (69) and (73) it is sufficient to show that

$$||R_{ij}||_p \leq r_{ij} \tag{77}$$

for all $i, j = 1, 2$. First, using the positivity of c_{++} etc we have

$$\begin{aligned} ||R_{11}||_p &= ||c_{++}X + c_{-+}Z||_p \leq c_{++}x' + c_{-+}z' = r_{11} \\ ||R_{22}||_p &= ||c_{+-}X + c_{--}Z||_p \leq c_{+-}x' + c_{--}z' = r_{22} \end{aligned}$$

The remaining bound also follows easily, since

$$\begin{aligned}
||R_{12}||_p &= ||t_1 \frac{(X+Z)}{2} + \lambda_1 Y_1 - i\lambda_2 Y_2||_p \\
&\leq ||t_1 \frac{(X+Z)}{2}||_p + ||(\lambda_1 - \lambda_2)Y_1 + \lambda_2(Y_1 - iY_2)||_p \\
&\leq t_1(x' + z')/2 + (\lambda_1 - \lambda_2)||Y_1||_p + \lambda_2||Y_1 - iY_2||_p
\end{aligned} \tag{78}$$

where in the last line we used (53). Furthermore

$$\begin{aligned}
||Y_1||_p &= ||(Y_1 - iY_2)/2 + (Y_1 + iY_2)/2||_p \\
&\leq \frac{1}{2}||Y_1 - iY_2||_p + \frac{1}{2}||Y_1 + iY_2||_p \\
&= y'
\end{aligned}$$

Hence (78) becomes

$$||R_{12}||_p \leq t_1(x' + z')/2 + (\lambda_1 - \lambda_2)y' + \lambda_2 y' = t_1(x' + z')/2 + \lambda_1 y' = r_{12} \tag{79}$$

which establishes the result.

7 Proofs of Corollaries

7.1 Corollary 5

Let Ω be any completely positive map on $\mathcal{B}(\mathbf{C}^K)$, and let τ be a state on $\mathcal{B}(\mathbf{C}^K \otimes \mathbf{C}^2)$ of the form

$$\tau = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \tag{80}$$

where A, B, C are $K \times K$ matrices, with $A \geq 0$, $C \geq 0$ and $\mathrm{Tr}(A+C) = 1$. Then $M = (\Omega \otimes I)(\rho)$ has the form (10) with $X = \Omega(A)$, $Y = \Omega(B)$ and $Z = \Omega(C)$. Hence from the definition of the maximal l_p norm it follows that

$$||X||_p \leq \nu_p(\Omega)\ \mathrm{Tr}(A), \quad ||Z||_p \leq \nu_p(\Omega)\ \mathrm{Tr}(C) \tag{81}$$

Applying (11) and using the facts that $(I \otimes \Phi)(M) = (\Omega \otimes \Phi)(\rho)$ and $\mathrm{Tr}(A) + \mathrm{Tr}(C) = \mathrm{Tr}(\rho) = 1$ we immediately deduce Corollary 5.

7.2 Corollary 2

Recall that the entropy of a state ρ is defined by

$$S(\rho) = -\mathrm{Tr}\rho \, \log \rho \tag{82}$$

Using $\mathrm{Tr}\rho = 1$ it follows that

$$\frac{d}{dp}\Big(||\rho||_p\Big)_{p=1} = -S(\rho), \tag{83}$$

and hence that

$$\frac{d}{dp}\Big(\nu_p(\Phi)\Big)_{p=1} = -S_{\min}(\Phi) \tag{84}$$

Therefore taking the derivative of (6) at $p = 1$ yields immediately (13).

7.3 Corollary 3

From the results of Section 2, it is sufficient to establish the bound (30). For any states ω and ρ we have

$$\log(\omega \otimes \rho) = \log \omega \otimes I + I \otimes \log \rho \tag{85}$$

Furthermore since Φ is a unital qubit map it follows that its optimal average output state is

$$\rho_\Phi = \frac{1}{2}\, I \tag{86}$$

Since $\log(\frac{1}{2}\, I) = -\log(2) I$ and $\mathrm{Tr}(\Omega \otimes \Phi)(\rho) = 1$ it follows that the left side of (30) can be written as

$$-\, S\Big((\Omega \otimes \Phi)(\tau)\Big) - \mathrm{Tr}\Big((\Omega \otimes \Phi)(\tau)\, \log(\rho_\Omega) \otimes I\Big) + \log(2) \tag{87}$$

Define

$$\omega = \mathrm{Tr}_2 \tau \tag{88}$$

Then the second term in (87) is equal to

$$-\, \mathrm{Tr}\, \Omega(\omega)\, \log(\rho_\Omega) \tag{89}$$

Also, the fact that Φ is unital implies that

$$\chi^*(\Phi) = \log(2) - S_{\min}(\Phi) \tag{90}$$

Hence to prove (30) it is sufficient to prove that

$$-S((\Omega \otimes \Phi)(\tau)) - \mathrm{Tr}\,\Omega(\omega)\,\log(\rho_\Omega) \leq \chi^*(\Omega) - S_{\min}(\Phi) \tag{91}$$

Now we use the bound (20), which is implied by (11). Again let τ have the form (80), so that $M = (\Omega \otimes I)(\tau)$ has the form (10) with $X = \Omega(A)$ and $Z = \Omega(C)$. Let $a = \mathrm{Tr}A = \mathrm{Tr}X$, and define the states

$$\alpha = \frac{1}{\mathrm{Tr}A}\,A = \frac{1}{a}\,A, \qquad \gamma = \frac{1}{\mathrm{Tr}C}\,C = \frac{1}{1-a}\,C \tag{92}$$

Then using the notation of (19), $\xi = \Omega(\alpha)$ and $\zeta = \Omega(\gamma)$, and (20) can be written

$$S((\Omega \otimes \Phi)(\tau)) \geq S_{\min}(\Phi) + aS(\Omega(\alpha)) + (1-a)S(\Omega(\gamma)) \tag{93}$$

Comparing with (91), it is sufficient to prove that

$$-aS(\Omega(\alpha)) - (1-a)S(\Omega(\gamma)) - \mathrm{Tr}\,\Omega(\omega)\,\log(\rho_\Omega) \leq \chi^*(\Omega) \tag{94}$$

Since $\omega = a\alpha + (1-a)\gamma$, we can rewrite the left side of (94) as

$$aS\Big(\Omega(\alpha)\,|\,\rho_\Omega\Big) + (1-a)S\Big(\Omega(\gamma)\,|\,\rho_\Omega\Big) \tag{95}$$

Since ρ_Ω is the optimal output state for the channel Ω, it follows from (24) that

$$S\Big(\Omega(\alpha)\,|\,\rho_\Omega\Big) \;\leq\; \chi^*(\Omega) \tag{96}$$

$$S\Big(\Omega(\gamma)\,|\,\rho_\Omega\Big) \;\leq\; \chi^*(\Omega) \tag{97}$$

Combining (94), (95) and (96) yields the result.

8 Summary

The results in this paper all concern product channels of the form $\Omega \otimes \Phi$, where in every case Ω is an arbitrary channel. For these product channels we prove

a variety of results involving different measures of the purity of output states from the channel.

The first set of results apply when Φ is a CQ or QC channel. Recall that the CQ channel first maps an input state to a letter in a classical alphabet, and then maps this to a quantum state at the output. The QC channel measures the input state with some POVM, and assigns different results to orthogonal output states. In both cases we prove that the output state with maximal l_p norm is a product state, and also that the Holevo capacity is achieved on a product state. In other words, the maximal l_p norm of the product channel is multiplicative and the Holevo capacity is additive. These results were previously shown to be true in the case where Φ is the identity map (and the additivity of the Holevo capacity also when $\Omega = \Phi$).

The second set of results apply when Φ is a qubit map, that is a map on states in $\mathbf{C}^2$. We prove multiplicativity for the $p = 2$ norm, for any qubit map Φ. We also prove multiplicativity for the l_p norm when p is any integer, and with some restrictions on Φ. The class of maps Φ satisfying the restrictions includes all unital qubit maps.

The third set of results revolves around a conjectured bound (11) for the l_p norm of any output state from the half-noisy channel $I \otimes \Phi$, when Φ is a qubit channel. We show that this bound implies multiplicativity of the l_p norm for any product channel $\Omega \otimes \Phi$. We also show that when Φ is *unital* the bound implies additivity of the Holevo capacity of the product channel $\Omega \otimes \Phi$. Therefore we believe that this conjecture provides a new and useful approach to the conjecture that the Holevo capacity is universally additive. In a hopeful sign of future progress on this important problem, Lieb and Ruskai have established Conjecture 4 in one non-trivial case (their proof appears as the Appendix below).

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A Appendix: Theorem of Lieb and Ruskai

Let $M = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$ and recall that M is positive semi-definite if and only $Y = \sqrt{X} R \sqrt{Z}$ with R a contraction. Moreover, any contraction can be written as a convex combination of unitary matrices. (See [10] or [15] for details and further references.) Hence, by the convexity of the p-norm, it suffices to prove (17) under the assumption that $Y = \sqrt{X} V \sqrt{Z}$ with V unitary.

We now consider the special case $X = Z$ and note that we can write

$$(I \otimes \Phi)(M) = \begin{pmatrix} X & \lambda Y \\ \lambda Y^* & X \end{pmatrix} = \sqrt{F} G \sqrt{F} \tag{98}$$

with $F = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$ and $G = \begin{pmatrix} I & \lambda V \\ \lambda V^* & I \end{pmatrix}$. We will use a result of Lieb and Thirring (Appendix B of [13]) that, for $p \geq 1$ and $F, G \geq 0$,

$$\mathrm{Tr}(F^{1/2} G F^{1/2})^p \leq \mathrm{Tr}(F^p G^p). \tag{99}$$

The critical feature is to note that G has eigenvalues $(1 \pm \lambda)$. Moreover,

$$\begin{pmatrix} I & \lambda V \\ \lambda V^* & I \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} I & V \\ V^* & -I \end{pmatrix} \begin{pmatrix} (1+\lambda) I & 0 \\ 0 & (1-\lambda) I \end{pmatrix} \begin{pmatrix} I & V \\ V^* & -I \end{pmatrix} \tag{100}$$

Thus

$$\begin{aligned} &\mathrm{Tr}[(I \otimes \Phi)(M)]^p \\ &\leq \tfrac{1}{2} \mathrm{Tr} \begin{pmatrix} I & V \\ V^* & -I \end{pmatrix} \begin{pmatrix} X^p & 0 \\ 0 & X^p \end{pmatrix} \begin{pmatrix} I & V \\ V^* & -I \end{pmatrix} \begin{pmatrix} (1+\lambda)^p I & 0 \\ 0 & (1-\lambda)^p I \end{pmatrix} \\ &= (1+\lambda)^p \, \mathrm{Tr} \tfrac{1}{2} (X^p + V X^p V^*) + (1-\lambda)^p \, \mathrm{Tr} \tfrac{1}{2} (X^p + V^* X^p V) \\ &= [2 \nu_p(\Phi)]^p \, \|X\|_p^p. \end{aligned}$$

Taking the p-th root gives the desired result, $\|(I \otimes \Phi)(M)\|_p \leq \nu_p(\Phi) \, 2 \|X\|_p$.

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